UDC 531.011

ON THE OPTIMIZATION OF SYSTEMS DEFINED BY STOCHASTIC DIFFERENTIAL EQUATIONS

PMM Vol 42, No. 5, 1978, pp 789-800

V. A. BODNER, N. E. RODNISHCHEV and E. P. IURIKOV
(Moscow, Kazan')

(Received January 3, 1978)

Discontinuous systems with optimizable discontinuity points defined by stochastic differential equations are considered. The principle of optimality of such systems is established as a natural extension of the principle of maximum, and a numerical method of determination of optimal control is proposed.

The considered problem allows important physical interpretations related, for instance, to the generalized control [1] which makes possible to consider control actions as instantaneous pulses, the selection of motion program [2] and of design parameters for composite aircraft, which ensure the highest probability of payload delivery to the specified destination, or the development of a programmed control of variable nature with the most precise realization of unperturbed motions, and others.

1. Statement of the problem. We have to determine the optimal control $v = (u, a, T)$ of a system whose behavior on the time interval $[t_0, t_k]$, which is successively defined on adjacent intervals $[t_{j-1}, t_j]$ $(j = 1,\ldots k)$ by the system of differential equations

$$dX_i(t) = \varphi_i^j(t, X, u, a)\,dt + \sum_{\nu=1}^{n} \sigma_{i\nu}^j(t, X)\,d\eta_\nu(t) \tag{1.1}$$

$$X_i(t_0) = X_{i0},\ t \in [t_{j-1}, t_j]\ (i = 1,\ldots, n;\ j = 1,\ldots, k)$$

that provides the minimum of functional

$$I_0(v) = \sum_{j=1}^{k} M\,[f_0(a, t_j X^j)] \tag{1.2}$$

with constraints

$$I_s(v) = \sum_{j=1}^{k} M\,[f_s(a, t_j, X^j)] = 0 \quad (s = 1,\ldots, q) \tag{1.3}$$

$$\psi_s(u) = 0\ (s = 1,\ldots, k_0),\ g_s(a) = 0\ (s = 1,\ldots, q_0)$$

where t is the time, $T = (t_0,\ldots, t_j,\ldots, t_k)$; with instants of time $t_0,\ldots, t_j,\ldots, t_k$ satisfy the inequalities $t_0 \leqslant \ldots \leqslant t_j \leqslant \ldots \leqslant t_k$, $X(t)$ is an n-dimensional vector function of state of the system that is continuous on every time interval $[t_{j-1}, t_j]$ $(j = 1, \ldots, k)$, $u(t)$ is an r—dimensional vector function of control, a is an m_0—dimensional determinate vector of control parameters, $\eta_\nu(t)$ are Wiener process independent in the aggregate, and $\varphi_i^j(t, X, u, a)$ and $\sigma_{i\nu}^j(t, X)$ are continuously differentiable functions determinate on $[t_{j-1}, t_j]$ that satisfy with unit probability the requirements for the existence of continuous solutions $X(t)$ and the probability density $p(t, X)$ of process $X(t)$ that satisfies the Kolmogorov —Fokker —Plank equation

$$\frac{\partial}{\partial t} p(t, X) = -\sum_{i=1}^{n} \frac{\partial}{\partial X_i} [\varphi_i^j(t, X, u, a) p(t, X)] + \tag{1.4}$$

$$\frac{1}{2} \sum_{i,\nu=1}^{n} \frac{\partial^2}{\partial X_i \partial X_\nu} [\sigma_{i\nu}^j(t, X) p(t, X)] = \Phi^j(t, p, u, a), \qquad t \in [t_{j-1}, t_j]$$

$$p(t, X)|_{t=t_0} = p(t_0, X)$$

where the superscript j defines the structure of the controlled system (1.1) on $[t_{j-1}, t_j]$, and $I_s(v)$ $(s = 0, \ldots, q)$ are continuous differentiable bounded functionals that represent mathematical expectation calculated by the formula

$$I_s(v) = \sum_{j=1}^{k} \int_{\Omega} f_s^j(a, t_j, X^j) p(t_j, X) \, dX \tag{1.5}$$

where $\Omega \subseteq E_n$ is the realization region of X. The symbol $I_s(v)$ emphasizes that $v = (u, a, T)$ are assumed to be independent variables. It is further assumed that the functional $I_0(v)$ is bounded below, i. e that $\inf I_0(v) = I_0(v^*) > -\infty$.

2. Necessary conditions of optimality. On above assumptions the problem (1.1) —(1.3) reduces similarly to that in [4] to the determinate problem of minimization of functional $I_0(v)$ (1.2) in the presence of the differential relation (1.4). The necessary conditions of optimality are defined by the following theorem.

Theorem 1 (the principle of optimality in the mean). Optimality of the admissible control $v^* = (u^*, a^*, T^*)$ of system (1.1) that results in the minimum of functional $I_0(v)$ requires the existence of a random function $\lambda(t, X) \not\equiv 0$ defined by the equations

$$\lambda_t = \frac{\partial \lambda(t, X)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial \lambda}{\partial X_i} \varphi_i^j(t, X, u, a) - \tag{2.1}$$

$$\frac{1}{2} \sum_{i,\nu=1}^{n} \frac{\partial^2 \lambda}{\partial X_i \partial X_\nu} \sigma_{i\nu}^j(t, X)$$

$$t \in [t_{j-1}, t_j] \quad (j = 1, \ldots, k)$$

$$\lambda(t_j, X)_- = \lambda(t_j, X)_+ - \sum_{s=0}^{q} \alpha_s f_s^{\,j}(a, t_j, X^j) \; (j = 1, \ldots, k-1) \tag{2.2}$$

$$\lambda(t_k, X)_- = -\sum_{s=0}^{q} \alpha_s f_s^{\,k}(a, t_k, X^k)$$

a) the optimal control u^* results in the maximum of function

$$M[R^j(t, X, u, a, \lambda)] = \int_{\Omega} R^j(t, X, u, a, \lambda)\, p(t, X)\, dX \tag{2.3}$$

$$R^j(t, X, u, a, \lambda) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial X_i} \varphi_i^{\,j}(t, X, u, a)$$

of variable u for almost all $t \in [t_{j-1}, t_j]$ $(j = 1, \ldots, k)$, and

b) parameters a^* and T^* satisfy the transversality condition

$$\sum_{j=1}^{k} \left(\sum_{s=0}^{q} \alpha_s M\left(\frac{\partial f_s^{\,j}}{\partial a}\right) - \int_{t_{j-1}^*}^{t_j^*} M\left(\frac{\partial R^j}{\partial a}\right) dt \right) + \sum_{s=1}^{q_0} \beta_s \frac{\partial g_s}{\partial a} = 0 \tag{2.4}$$

$$M(\lambda_t^{\,j} - \lambda_t^{\,j-1})_{t=t_j^*} + \sum_{s=0}^{q} \alpha_s M\left(\frac{\partial f_s^{\,j}}{\partial t_j}\right) = 0$$

Corollary. The optimal control u^* satisfies the relation

$$M\left(\frac{\partial R^j}{\partial u}\right) = \sum_{s=1}^{k_0} \mu_s(t) \frac{\partial \psi_s}{\partial u} \tag{2.5}$$

When u^* obtains in the interior of a set of controls, the right-hand side of (2.5) is zero.

3. Example. Let it be required to determine the optimal control u^* of a system whose behavior on the time interval $[0, t_2]$ is successively defined on adjacent intervals $[0, t_1]$ and $[t_1, t_2]$ by the stochastic equations

$$\begin{aligned} &dX_1 = X_2 dt, \; dX_2 = u dt + \sigma d\eta \\ &X_1(0) = c_1, \; X_2(0) = c_2; \; t \in [0, t_1] \\ &X_1(t_1) = X_1(t_1)_+, \; X_2(t_1)_- = X_2(t_1)_+; \; t \in [t_1, t_2] \end{aligned} \tag{3.1}$$

which provide the minimum of functional

$$I_0(u) = \sum_{j=1}^{2} M[X_1(t_j) - c_j]^2 \tag{3.2}$$

with constraint on the control

$$|u| \leqslant 1 \tag{3.3}$$

In conformity with the necessary conditions of optimality we seek the optimal control u^* using the condition

$$R(t, X, u^*, a, \lambda) = \max_u R(t, X, u, a, \lambda) \tag{3.4}$$
$$R(t, X, u, a, \lambda) = \frac{d\lambda}{\partial X_1} X_2 + \frac{\partial \lambda}{\partial X_2} u$$

where $\lambda(t, X)$ is determined by the solution of equation

$$\frac{\partial \lambda}{\partial t} = -L\lambda, \quad L = X_2 \frac{\partial}{\partial X_1} + u \frac{\partial}{\partial X_2} + \frac{\sigma}{2} \frac{\partial^2}{\partial X_1 \partial X_2} \tag{3.5}$$
$$\lambda(t_2, X)_- = -[X_1(t_2) - c_2]^2, \ t \in [t_2, t_1]$$
$$\lambda(t_1, X)_- = \lambda(t_1, X)_+ - [X_1(t_1) - c_1]^2, \ t \in [t_1, 0]$$

Using (3.4) and taking into account (3.3) we determine the optimal control

$$u^*(t) = \operatorname{sign} \frac{\partial \lambda}{\partial X_2} \tag{3.6}$$

Taking into consideration (3.6) and introducing the inverse time $\tau = t_2 - t$ from (3.5) we obtain

$$d\lambda/dt = L\lambda \tag{3.7}$$
$$\lambda(\tau_2, X) = -[X_1(\tau_2) - c_2]^2, \ \tau \in [\tau_2, \tau_1], \ \tau_2 = 0$$
$$\lambda(\tau_1, X)_- = \lambda(\tau_1, X)_+ - [X_1(\tau_1) - c_1]^2, \ \tau \in [\tau_1, \tau_0]$$

We seek the solution of $\lambda(\tau, X_1, X_2)$ of the linearly quadratic form [5] with indeterminate coefficients

$$\lambda = k_0(\tau) + k_1(\tau)X_1 + k_2(\tau)X_2 + k_{11}(\tau)X_1^2 + k_{12}(\tau)X_1X_2 + k_{22}(\tau)X_2^2 \tag{3.8}$$

Substituting the expressions for λ into (3.7) and equating coefficients at equal $X = (X_1, X_2)$ we obtain for $k(\tau)$ on $[\tau_2, \tau_1]$ the system of ordinary differential equations

$$k_0^{\cdot} = k_2 + \sigma k_{22}, \ k_1^{\cdot} = k_{12}, \ k_2^{\cdot} = k_1 + 2k_{12} \tag{3.9}$$
$$k_{11}^{\cdot} = 0, \ k_{12}^{\cdot} = 2k_{11}, \ k_{22}^{\cdot} = k_{12}$$

with initial conditions

$$k_0(\tau_2) = k_2(\tau_2) = k_{12}(\tau_2) = k_{22}(\tau_2) = 0, \ k_1(\tau_2) = 2c_2, \ k_{11}(\tau_2) = -1$$

Solving (3.9) with initial conditions in inverse time we obtain

$$k_0(\tau) = c_2\tau^2 - \frac{\sigma}{3}\tau^3 - \frac{1}{4}\tau^4, \quad k_1(\tau) = 2c_2 - \tau^2, \quad k_2(\tau) = 2c_2\tau - \tau^3$$

$$k_{11}(\tau) = -1, \ k_{12}(\tau) = -2\tau, \ k_{22}(\tau) = -\tau^2$$

As the result we can represent the optimal synthesizing function u^* on $[\tau_2, \tau_1]$ in the form

$$u^*(\tau) = \text{sign}\,[2c_2\tau - \tau^3 - 2\tau X_1 - 2\tau^2 X_2] \tag{3.10}$$

On $[\tau_1, \tau_0]$ the boundary conditions of system (3.9) are

$$\begin{aligned} &k_0(\tau_1) = 0, \ k_1(\tau_1) = 2(c_1 + c_2) - \tau_1^2, \ k_2(\tau_1) = 2c_2\tau_1 - \tau_1^3 \\ &k_{11}(\tau_1) = -2, \ k_{12}(\tau_1) = -\tau_1, \ k_{22}(\tau_1) = -\tau_1^2 \end{aligned} \tag{3.11}$$

Solving (3.9) in inverse time with boundary conditions (3.11) on $[\tau_1, \tau_0]$, we obtain the optimal synthesizing function u^* on $[\tau_1, \tau_0]$ of the form

$$\begin{aligned} u^*(\tau) = \text{sign}\,[&{}^1/_2\tau_1^2 - 2c_1\tau_1 + 2(c_1 + c_2)\tau - {}^3/_2\tau_1\tau^2 - 2(\tau - \tau_1)^3 + \\ &(-\tau_1 - 4(\tau - \tau_1))X_1 + 2(-\tau_1\tau - 2(\tau - \tau_1)^2)X_2] \end{aligned} \tag{3.12}$$

It follows from (3.10) and (3.12) that the optimal control is a piece-wise-constant function whose values are ± 1.

4. Numerical method of optimal control search. For the numerical determination of optimal control $v^* = (u^*, a^*, T^*)$ of problem (1.1)–(1.3) we apply the method of gradient projection. The computation algorithm for that method is determined by the form of the formula of the Lagrange first variation functional for arbitrary control variations. Let us compose for problem (1.1)–(1.3) the Lagrange functional

$$\begin{aligned} F(v, \eta) = I_0(v) + \sum_{s=1}^{q} \alpha_s I_s(v) + \\ \sum_{s=1}^{k_0} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \mu_s(t)\psi_s(u)\,dt + \sum_{s=1}^{q_0} \beta_s g_s(a) \end{aligned} \tag{4.1}$$

where $\eta = (\alpha, \mu(t), \beta)$ is the vector of Lagrange, multipliers. We introduce the random function $\lambda^{js}(t, X)$, $(j = 1, \ldots, k;\ s = 0, \ldots, q)$ which is continuous on $[t_{j-1}, t_j]$ and satisfies the equation

$$\begin{aligned} \lambda_t^{js} = \frac{\partial \lambda^{js}}{\partial t} = -\sum_{i=1}^{n} \frac{\partial \lambda^{js}}{\partial X_i}\varphi_i^j(t, X, u, a) - \\ \frac{1}{2}\sum_{i,\nu=1}^{n} \frac{\partial^2 \lambda^{js}}{\partial X_i \partial X_\nu}\sigma_{i\nu}^j(t, X) \end{aligned} \tag{4.2}$$

$$t \in [t_j,\ t_{j-1}] \quad (j = 1, \ldots, k)$$
$$\lambda^{js}(t_j, X)_- = \lambda^{js}(t_j, X)_+ - f_s^j(a, t_j, X^j) \quad (j = 1, \ldots, k-1)$$
$$\lambda^{ks}(t_k, X)_- = -f_s^k(a, t_t, X^k)$$

The formula for the first variation of the Lagrange functional can now be represented in the form

$$\delta F = \frac{\partial}{\partial \varepsilon} F(v + \varepsilon\delta v, \eta) = \tag{4.3}$$
$$-\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big[ML_u R^{js} - \sum_{s=0}^{k_s} \mu_s(t) \frac{\partial \psi_s}{\partial u}\Big] \delta u\, dt +$$
$$\Big[\sum_{j=1}^{k} M\Big(L_a f_s^j - \int_{t_{j-1}}^{t_j} L_a R^{js}\Big) + \sum_{s=1}^{q} \beta_s \frac{\partial g_s}{\partial a}\Big] \delta a +$$
$$\sum_{j=1}^{k} [ML_t(\lambda^{js} - \lambda^{j-1,s})|_{t=t_j} + L_{t_j} f_s^j]\, \delta t_j$$
$$R^{js} = \sum_{i=1}^{n} \frac{\partial \chi^{js}}{\partial X_i} \varphi_i^j(t, X, u, a) \quad (j = 1, \ldots, k;\ s = 0, \ldots, q)$$
$$\alpha_0 = 1, \quad L_u = \sum_{s=0}^{q} \alpha_s \frac{\partial}{\partial u}, \quad L_a = \sum_{s=0}^{q} \alpha_s \frac{\partial}{\partial a} \quad \text{etc.}$$

In conformity with the concept of the gradient method it is reasonable to consider in the functional space the following iterative process of search for the optimal control:

$$u_\rho^{n+1} = u_\rho^n + h^n [ML_{u_\rho} R^{js} - L^{u_\rho} \psi_s] \quad (\rho = 1, \ldots, r;\ j = 1, \ldots, k) \tag{4.4}$$
$$a_i^{n+1} = a_i^n - h^n \Big[\sum_{j=1}^{k} M\Big(L_{a_i} f_s^j - \int_{t_{j-1}}^{t_j} L_{a_i} R^{is}\, dt\Big) + L^{a_i} g_s\Big]$$
$$(i = 1, \ldots, m_0)$$
$$t_j^{n+1} = t_j^n - h^n [ML_t(\lambda^{js} - \lambda^{j-1,s})|_{t=t_j} + ML_{t_j} f_s^j] \quad (j = 1, \ldots, k)$$

where n is the ordinal number of iteration, h^n is the pitch of the n-th iteration. Expressions in the right-hand sides of (4.4) are calculated using the n-th solutions of (1.4) and (4.2) [6, 7].

To determine multipliers $\alpha_\varepsilon, \mu_s(t)$ and β_s we select from all directions $\delta u_\rho^n(t)$, δa_i^n, and δt_j^n defined by the respective expressions in brackets in the right-hand sides of (4.4) the admissible directions that satisfy in conformity with the method of gradient projection the conditions

$$\delta I_m = I_m'(v)\,\delta v^n = -\sum_{\rho=1}^{r}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} M\left(\frac{\partial R^{jm}}{\partial u_\rho}\right)\delta u_\rho^n(t)\,dt + \tag{4.5}$$

$$\sum_{i=1}^{m_0}\sum_{j=1}^{k} M\left(\frac{\partial f_m^{\,j}}{\partial a_i} - \int_{t_{j-1}}^{t_j}\frac{\partial R^{jm}}{\partial a_i}\,dt\right)\delta a_i^n +$$

$$\sum_{j=1}^{k} M\left((\lambda_t^{jm} - \lambda_t^{j-1,\,m})\,|_{t=t_j} + \frac{\partial f_m^{\,j}}{\partial t_j}\right)\delta t^n{}_j = 0 \quad (m = 1, \ldots, q)$$

$$\psi_m'(u)\,\delta u^n(t) = \sum_{\rho=1}^{r}\frac{\partial\psi_m}{\partial u_\rho}\,\delta u_\rho^n(t) = 0 \quad (m = 1, \ldots, k_0)$$

$$g_m'(a)\,\delta a^n = \sum_{i=1}^{m_0}\frac{\partial g_m}{\partial a_i}\,\delta a_i^n = 0 \quad (m = 1, \ldots, q_0)$$

Substituting $\delta u_\rho^n(t)$, δa_i^n, and δt_j^n determined by expressions in brackets in (4.4) and (4.5), we obtain a closed system of linear equations in the unknown Lagrange multipliers $\alpha_s, \mu_s(t)$, and β_s. Since $u(t)$ has an infinite number of values for every t, it is evident that system (4.5) consists of an infinite number of equations. To overcome this difficulty we cover the time interval $[t_{j-1}, t_j]$ $(j = 1, \ldots, k)$ by a denumerable net of pitch τ with nodes $t_\nu = \nu\tau$ $(\nu = 1, \ldots, N; t_j = N\tau; \nu \neq j)$, and carry out control function refinements with the required degree of accuracy at each fixed $t_\nu \in [t_{j-1}, t_j]$.

Thus the iteration process of optimal control determination reduces to the following:

$$u_\rho^{n+1}(t_\nu) = u_\rho^n(t_\nu) + h^n\,[ML_{u_\rho}\;R^{js} - L^{u_\rho}\psi_s]_{t=t_\nu} \quad (\rho = 1, \ldots, r;\ j = 1, \ldots, k;\ \nu = 1, \ldots, N) \tag{4.6}$$

$$a_i^{n+1} = a_i^n - h^n\Big[\sum_{j=1}^{k} M\,(L_{a_i}(f_s^{\,j} - \tau R^{js}\,|_{t=t_\nu})) + L^{a_i}g_s\Big] \quad (i = 1, \ldots, m_0)$$

$$t_j^{n+1} = t_j^n - h^n\,[ML_t\,(\lambda^{js} - \lambda^{j-1,\,s})\,|_{t=t_j} + L_{t_j}f_s^{\,j}] \quad (j = 1, \ldots, k)$$

where derivatives in the right-hand sides are calculated for specified $u_\rho^n(t)$, a_i^n and t_j^n at each fixed point of the cylinder $\Omega \times [t_0, t_k]$ along the trajectories (1.4) and (4.2). Multipliers α_s, $\mu_s(t)$, and β_s are determined by solving the nondegenerate system of linear algebraic equations

$$\sum_{s=1}^{q}\alpha_s B_{ms} + \sum_{s=q+1}^{q+k_0}\sum_{\nu=1}^{N}\mu_s(t_\nu)\,B_{ms} + \sum_{s=q+k_0+1}^{q+k_0+q_0}\beta_s B_{ms} = B_{m0} \quad (m = 1, \ldots, q) \tag{4.7}$$

$$\sum_{s=1}^{q}\alpha_s C_{ms} + \sum_{s=q+1}^{q+k_0}\mu_s(t_\nu)\,C_{ms} = C_{m0} \quad (m = 1, \ldots, k_0)$$

$$\sum_{s=1}^{q} \alpha_s D_{ms} + \sum_{s=q+k_0+1}^{q+k_0+q_0} \beta_s D_{ms} = D_{m0} \quad (m = 1, \ldots, q_0)$$

Coefficients at the unknown α_s, $\beta_s(t)$, and λ_s are defined by the equations

$$B_{ms} = \begin{cases} (-1)^l \sum_{i=1}^{m_0} A_{im}A_{is} + \sum_{j=1}^{k} Q_{jm}Q_{js} - \sum_{\nu=1}^{N} U_{\nu ms}, \\ s = \begin{cases} 1, \ldots, q; & l = 0 \\ 0; & l = 1 \end{cases} \\ U_{\nu ms}, \quad s = q+1, \ldots, q+k_0 \\ \sum_{i=1}^{m_0} A_{im} \frac{\partial g_s}{\partial a_i}, \quad s = q+k_0+1, \ldots, q+k_0+q_0 \end{cases}$$

$$C_{ms} = \begin{cases} (-1)^l \sum_{\rho=1}^{r} M\left(\frac{\partial \psi_m}{\partial u_\rho} \frac{\partial R^{js}}{\partial u_\rho}\right), & s = \begin{cases} 1, \ldots, q, & l = 0 \\ 0, & l = 1 \end{cases} \\ \sum_{\rho=1}^{r} \left(\frac{\partial \psi_m}{\partial u_\rho} \frac{\partial \psi_s}{\partial u_\rho}\right), & s = q+1, \ldots, q+k_0 \end{cases}$$

$$D_{ms} = \begin{cases} (-1)^l \sum_{i=1}^{m_0} \frac{\partial g_m}{\partial a_i} A_{is} & s = \begin{cases} 1, \ldots, q; & l = 0 \\ 0, & l = 1 \end{cases} \\ \sum_{i=1}^{m_0} \frac{\partial g_m}{\partial a_i} \frac{\partial g_s}{\partial a_i}, & s = q+k_0+1, \ldots, q+k_0+q_0 \end{cases}$$

$$A_{im} = \sum_{j=1}^{k} M\left(\frac{\partial f_m^j}{\partial a_i} - \tau \sum_{\nu=1}^{N} \frac{\partial R^{jm}}{\partial a_i}\bigg|_{t=t_\nu}\right), \quad A_{is} = A_{im}|_{m=s}$$

$$Q_{jm} = M\left((\lambda_t^{jm} - \lambda_t^{j-1,m})|_{t=t_j} + \frac{\partial f_m^j}{\partial t_j}\right), \quad Q_{js} = Q_{jm}|_{m=s}$$

$$U_{\nu ms} = \tau \sum_{\rho=1}^{r} \sum_{j=1}^{k} M\left(\frac{\partial R^{jm}}{\partial u_\rho} \frac{\partial R^{js}}{\partial u_\rho}\right)\bigg|_{t=t_\nu}$$

Note that the optimal control obtained for the determinate problem formulated on the assumption that the system is free of interference, i.e. that $\sigma_{i\nu}^j(t, X) = 0$, can be taken as the initial approximation $u_\rho^\circ(t)$, a_i°, t_j° of the stochastic problem.

Convergence of the process to the optimal solution can be proved using the reasoning of the proof in [5] with some additional assumptions.

The general form of the gradient iteration process (4.6) is

$$v^{n+1} = v^n - h^n F_v'(v^n, \eta^n) \qquad (4.8)$$

where $\eta^n = (\alpha^n, \mu^n(t), \beta^n)$ satisfy the nondegenerate system of linear equations (4.7).

Theorem 2. If the functional $F(v, \eta)$ is bounded below with respect to v and the gradient $F_v'(v, \eta)$ satisfies the Lipschitz condition with constant M_0, $0 < \varepsilon_1 \leqslant h^n \leqslant 2(M_0 + 2\varepsilon_2)$, $\varepsilon_2 > 0$, then the following statements hold for the sequence (4.8).

1°. The functional $F_v'(v^n, \eta^n)$ monotonically decreases with v, and $\lim \| v^{n+1} - v^n \| = 0$ when $n \to \infty$.

2°. $\lim F_v'(v^n, \eta^n) = 0$, when $n \to \infty$.

3°. If the functional $F(v, \eta)$ is convex with respect to v, then

$$\lim F(v^n, \eta^n) = F(v^*, \eta) = \inf_v F(v, \eta), \quad n \to \infty$$

5. Example. Let us consider the problem of finding the optimal control for system (3.1) which ensures the minimum of functional

$$I_0(u) = M[X_1(t_2)]$$

with the constraint

$$I_1(u) = M[X_2(t_2) - c_2] = 0$$

In conformity with the described method the optimal control (u^*, t_1^*) is determined by the following iteration procedure:

$$u^{n+1}(t_\nu) = u^n(t_\nu) - h^n M\left(\frac{\partial R^{10}}{\partial u} + \alpha\frac{\partial R^{11}}{\partial u}\right)\Big|_{t=t_\nu}, \quad t_\nu \in [0, t_1] \tag{5.1}$$

$$u^{n+1}(t_\nu) = u^n(t_\nu) - h^n M\left(\frac{\partial R^{20}}{\partial u} + \alpha\frac{\partial R^{21}}{\partial u}\right)\Big|_{t=t_\nu}, \quad t_\nu \in [t_1, t_2]$$

$$t_1^{n+1} = t_1^n + h^n M[(\lambda_t^{10})|_{t=t_1^n} + \alpha(\lambda_t^{11})|_{t=t_1^n}]$$

$$\alpha = \left\{\tau\sum_{\nu=1}^{N}\sum_{j=1}^{2} M\left(\frac{\partial R^{j1}}{\partial u}\frac{\partial R^{j0}}{\partial u}\right)_{t=t_\nu} - \sum_{j=1}^{2} M(\lambda_t^{j1} - \lambda_t^{j-1,1})(\lambda_t^{j0} - \lambda_t^{j-1,0})|_{t=t_1^n}\right\}\times$$
$$\left\{\sum_{j=1}^{2} M(\lambda_t^{j1} - \lambda_t^{j-1,1})^2|_{t=t_1^n} - \tau\sum_{\nu=1}^{N}\sum_{j=1}^{2} M\left(\frac{\partial R^{j1}}{\partial u}\right)^2\Big|_{t=t_\nu}\right\}^{-1}$$

where expressions in the right-hand sides are determined by the n-th solutions of equations

$$L^- p = 0; \quad p(t, X)|_{t=t_0} = p(t_0, X), \quad t \in [0, t_1] \tag{5.2}$$

$$p(t, X)|_{t=t_1} = p(t_1, X), \quad t \in [t_1, t_2]$$

$$L^+\lambda^{20} = 0, \quad L^+\lambda^{21} = 0, \ \lambda^{20}(t_2) = -X_1(t_2)$$

$t \in [t_2, t_1]$; $\lambda^{21}(t_2)_- = -[X_2(t_2) - c_2]$

$L^+\lambda^{10} = 0, \quad L^+\lambda^{11} = 0, \quad \lambda^{10}(t_1)_- = \lambda^{10}(t_1)_+; \; \lambda^{11}(t_1)_- = \lambda^{11}(t_1)_+$

$t \in [t_1, 0]$

$$L^{\pm} = \frac{\partial}{\partial t} + X_2\frac{\partial}{\partial X_1} + u\frac{\partial}{\partial X_2} \pm \frac{\partial^2}{\partial X_2 \partial X_2}$$

The expressions for R^{j0} and R^{j1} $(j = 1, 2)$ are of the form

$$R^{j0} = X_2\frac{\partial\lambda^{j0}}{\partial X_1} + u\frac{\partial\lambda^{i0}}{\partial X_2}, \quad R^{j1} = X_2\frac{\partial\lambda^{j1}}{\partial X_1} + u\frac{\partial\lambda^{j1}}{\partial X_2}$$

6. Proof of Theorem 1. To prove the optimality conditions we use the principle of Lagrange [9]. For the problem of minimization of functional $I_0(v)$ with the differential relation (1.4) and constraint (1.3) we define the Lagrange functional in the form (4.1). Let $v^* = (u^*, a^*, T^*)$ be the optimal control. Let us consider the control $v = (u, a, T)$, where

$$u(t) = u^*(t) + \Delta u(t),$$

$$a = a^* = \varepsilon\delta a, \quad T = T^* + \varepsilon\delta T$$

$$\Delta u(t) = \begin{cases} \varepsilon\delta u(t), & t \in [t_0, t_k] \setminus U_i y_i \\ \omega - u^*, & t \in y_i \end{cases}$$

where y_i $(i = 1, \ldots, k)$ is a half-open interval over which an almost impulsive variation $\delta u = \omega_i - u^*$ occurs [10] defined on $\tau_i \leqslant t \leqslant \tau_i + \varepsilon l_i$.

Owing to the continuous differentiability of functions $\varphi_i^j(t, X, u, a), \delta_{iv}{}^j(t, X)$ with respect to t, X, u, and a, solution of (1.4) with initial $p(t_0, X)$ is continuously differentiable with respect to ε, when $\Delta u(t)$, δa, and δT are fixed. We denote that solution by $p_\varepsilon(t, X_\varepsilon, \Delta u, \delta a, \delta T)$ and determine the variation

$$\delta p = \lim_{\varepsilon \to 0} \frac{p_\varepsilon - p}{\varepsilon} = \left.\frac{\partial p_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}$$

which specifically means that when $\varepsilon \to 0$ the term $\varepsilon\delta p$ is the principal linear part of $p + \varepsilon\delta p$ generated by the variation of control.

Since $v^* = (u^*, a^*, T^*)$ is the optimal control, the first variation of the Lagrange functional

$$\delta F(v, \eta) = \frac{\partial}{\partial \varepsilon} F(v^* + \varepsilon\delta v, \eta) = \tag{6.1}$$

$$\sum_{s=0}^{q}\alpha_s\sum_{j=1}^{k}\left[\int_{\Omega} f_s^j\delta p\,dX + M\left(\frac{\partial f_s^j}{\partial t_j}\right)\delta t_j + M\left(\frac{\partial f_s^j}{\partial a}\right)\delta a\right] +$$
$$\sum_{s=1}^{k_0}\sum_{j=1}^{k}\int_{t_{j-1}^*}^{t_j^*}\mu_s(t)\frac{\partial\psi_s}{\partial u}\delta u\,dt + \sum_{s=1}^{q_0}\beta_s\frac{\partial g_s}{\partial a}\delta a \geqslant 0$$

must be nonnegative.

Let us determine the first derivative of (6.1). From (1.4) we have

$$\frac{\partial}{\partial\varepsilon}[p_\varepsilon - p] = \Phi^i(t, p_\varepsilon, u^* + \Delta u, a^* + \varepsilon\delta a) - \Phi^i(t, p, u^*, a^*) \qquad (6.2)$$

Multiplying both sides of (6.2) by the indeterminate random function $\lambda(t, X) \not\equiv 0$ and integrating over the cylinder $\Omega \times [t_0^*, t_k^*]$ taking into account that $u = u^* + \Delta u = \omega$ when $t \in [\tau, \tau + \varepsilon l]$, after dividing by ε and passing to limit with allowance for the theorem on finite increments, we obtain

$$\sum_{j=1}^{k}\int_{t_{j-1}^*}^{t_j^*}\int_{\Omega}\lambda\frac{\partial(\delta p)}{\partial t}dX\,dt = \qquad (6.3)$$
$$\sum_{j=1}^{k}\int_{t_{j-1}^*}^{t_j^*}\int_{\Omega}\lambda\left[\frac{\partial\Phi^j}{\partial p}\delta p + \frac{\partial\Phi^j}{\partial u}\delta u + \frac{\partial\Phi^j}{\partial a}\delta a\right]dX\,dt +$$
$$\sum_{j=1}^{k}\int_{\Omega}[\Phi^j(t, p, \omega, a^*) - \Phi^j(t, p, \omega^*, a^*)]\,|_{t=\tau}\, l\,dX + O(\varepsilon)$$

Without the loss of generality of results we shall consider a single needle-like variation, since owing to the additivity properties of (6.3) the effect of several needle-like variations occurring over various infinitely short time intervals can be considered independently of each other. Integrating by parts the left-hand side of (6.3) and writing the expression in the right-hand side with allowance for (2.1) and (2.3) and transformation of integrals [4], we obtain

$$\sum_{j=1}^{k}\int_{\Omega}[\lambda(t_j^*, X)\,\delta p(t_j^*, X) - \lambda(t_{j-1}^*, X)\,\delta p(t_{j-1}^*, X)]\,dX = \qquad (6.4)$$
$$\sum_{j=1}^{k}\int_{t_{j-1}^*}^{t_j^*}\int_{\Omega}\left[\frac{\partial R^j}{\partial u}\delta u + \frac{\partial R^j}{\partial a}\delta a\right]p\,dX\,dt +$$

$$\sum_{j=1}^{k}\int_{\Omega}[R^j(t,X,\omega,a^*,\lambda)-R^j(t,X,u^*,a^*,\lambda)]\,|_{t=\tau}\,lp\,dX+O(\varepsilon)$$

Let us determine variations $\delta p\ (t_j{}^*,\ X)$, and $\delta p\ (t_{j-1}{}^*,\ X)$. On the strength of the supplementary definition of $u\ (t)$ we pass to the integral form (1.4) beyond the limits $[t_{j-1}{}^*,\ t_j{}^*]$, we find that on the fairly short interval $[t_j{}^*,\ t_j]$

$$p(t_j{}^*,x)=p(t_j,X)-\int_{t_j^*}^{t_j}p_t\,dt,\quad t_j=t_j{}^*+\varepsilon\delta t_j \tag{6.5}$$

Using the theorem on finite increments and taking into account that $\delta p_j = p\ (t_j,\ X) - p\ (t_j{}^*, X^*)$, and $\delta p\ (t_j{}^*,\ X) = p\ (t_j{}^*,\ X) - p\ (t_j{}^*,\ X^*)$, from (6.5) we obtain

$$\delta p\ (t_j{}^*,\ X)\ =\ \delta p_j-(p_t\,|_{t=t_j^*})\ \delta t_j+O\ (\varepsilon) \tag{6.6}$$

Similary we obtain

$$\delta p\ (t_{j-1}{}^*,X)\ =\ \delta p_{j-1}-(p_t\,|_{t=t_{j-1}^*})\delta t_{j-1}+O\ (\varepsilon) \tag{6.7}$$

The substitution of (6.6) and (6.7) into (6.4) yields

$$\begin{aligned}&\int_{\Omega}\Big\{\sum_{j=1}^{k-1}[\lambda(t_j{}^*,X)_- -\lambda(t_j{}^*,X)_+]\,\delta p_j+\lambda(t_k{}^*,X)\,\delta p_k+\\&\sum_{j=1}^{k}(\lambda_t{}^j-\lambda_t^{j-1})_{t=t_j^*}p\delta t_j\Big\}\,dX=\\&\sum_{j=1}^{k}\int_{t_{j-1}^*}^{t_j^*}\int_{\Omega}\Big[\frac{\partial R^j}{\partial u}\,\delta u+\frac{\partial R^j}{\partial a}\,\delta a\Big]p\,dX\,dt+\\&\sum_{j=1}^{k}\int_{\Omega}[R^j(t,X,\omega,a^*,\lambda)-R^j(t,X,\omega^*,a^*,\lambda)]\,|_{t=\tau}lp\,dx+O(\varepsilon)\end{aligned} \tag{6.8}$$

We use formulas (4.2) and (4.3) for determining the random functions $\lambda^s\ (t,\ X)$ and $R^{js}\ (t,\ X,\ u,\ a,\ \lambda)$ appearing in (6.8). Taking into account (6.8) from (6.1) we have

$$\delta F(v,\eta)=\sum_{j=1}^{k}\int_{t_{j-1}^*}^{t_j^*}\Big[-\sum_{s=0}^{q}\alpha_s M\Big(\frac{\partial R^{js}}{\partial u}\Big)+\sum_{s=1}^{k_0}\mu_s(t)\frac{\partial\psi_s}{\partial u}\Big]\delta u\,dt+ \tag{6.9}$$
$$\Big[\sum_{s=0}^{q}\alpha_s\sum_{j=1}^{k}M\Big(\frac{\partial f_s^j}{\partial a}-\int_{t_{j-1}^*}^{t_j^*}\frac{\partial R^{js}}{\partial a}\,dt\Big)+\sum_{s=1}^{q_0}\beta_s\frac{\partial g_s}{\partial a}\Big]\delta a+$$
$$\sum_{j=1}^{k}\sum_{s=0}^{q}\alpha_s M\Big[(\lambda_t^{js}-\lambda_t^{j-1,s})\big|_{t=t_j^*}+\frac{\partial f_s^j}{\partial t_j}\Big]\delta t_j-$$
$$\sum_{s=0}^{q}\alpha_s\sum_{j=1}^{k}M\,[R^{js}(t,X,\omega,a^*,\lambda)-R^{js}(t,X,u^*,a^*,\lambda)]\big|_{t=\tau}\,l\geqslant 0$$

which shows that for any selection of variations δu, δa, and δT the following necessary optimality conditions:

$$M\Big(\frac{\partial R^j}{\partial u}\Big)-\sum_{s=1}^{k_0}\mu_s(t)\frac{\partial\psi_s}{\partial u}=0 \tag{6.10}$$
$$\sum_{j=1}^{k}\Big(\sum_{s=0}^{q}\alpha_s M\Big(\frac{\partial f_s^j}{\partial a}\Big)-\int_{t_{j-1}^*}^{t_j^*}M\Big(\frac{\partial R^j}{\partial a}\Big)dt\Big)+\sum_{s=1}^{q_0}\beta_s\frac{\partial g_s}{\partial a}=0$$
$$M(\lambda_t^j-\lambda_t^{j-1})\big|_{t=t_j^*}+\sum_{s=0}^{q}\alpha_s M\Big(\frac{\partial f_s^j}{\partial t_j}\Big)=0$$
$$M\,[R^j(t,X,\omega,a^*,\lambda)-R^j(t,X,u^*,a^*,\lambda)]\big|_{t=\tau}\leqslant 0$$

must be satisfied at point $\sigma^*=(u^*,a^*,T^*)$. Since the last inequality in (6.10) holds for any $\omega\in U$, hence

$$M\,[R^j(t,X,u^*,a^*,\lambda)]=\max_{\omega\in U}M\,[R^j(t,X,\omega,a^*,\lambda)]$$

from which follows statement a) of Theorem 1. Conditions (6.10) conform to conditions b) of that theorem and to its corollary (2.5).

7. Proof of Theorem 2. The conditions of continuity of $F_v'(v,\cdot)$ imply that

$$F(v^{n+1},\cdot)-F(v^n,\cdot)=-h^n\int_0^1 F_v'(v^n-th^nF_v'(v^n,\cdot),\cdot)\times$$

$$F_v'(v^n,\cdot)\,dt = -h^n\int_0^1 [F_v'(v^n,\cdot) - F_v'(v^n,\cdot) +$$
$$F_v'(v^n - th^nF_v'(v^n,\cdot),\cdot)]\,F_v'(v^n,\cdot)\,dt \leqslant -h^n\,\|F_v'(v^n,\cdot)\|^2 +$$
$$(h^n)^2\,\|F_v'(v^n,\cdot)\|^2 M\int_0^1 t\,dt = \left(\frac{(h^n)^2 M}{2} - h^n\right)\|F_v'(v^n,\cdot)\|^2 =$$
$$\left(\frac{M}{2} - \frac{1}{h^n}\right)\|v^{n+1} - v^n\|^2 \leqslant -\varepsilon_2\,\|v^{n+1} - v^n\|^2 \leqslant 0$$

Thus $F(v^n,\cdot)$ monotonically decreases with respect to v. Since the functional $F(v,\cdot)$ is bounded with respect to v, hence $\lim_{n\to\infty} F(v^n,\cdot)$ exists and consequently

$$\|v^{n+1} - v^n\|^2 \leqslant [F(v^n,\cdot) - F(v^{n+1},\cdot)]\,/\,\varepsilon_2 \to 0$$
$$\|F_v'(v^n,\cdot)\|^2 \leqslant \frac{F(v^n,\cdot) - F(v^{n+1},\cdot)}{(h^n)^2\,(1/h^n - M/2)} \leqslant \frac{F(v^n,\cdot) - F(v^{n+1},\cdot)}{\varepsilon_1^2\varepsilon_2} \to 0$$

when $n\to\infty$. The first two statements are proved.

If the functional $F(v,\cdot)$ is convex with respect to v, then

$$0 \leqslant F(v^n,\cdot) - F(v^*,\cdot) \leqslant F_v'(v^n,\cdot)(v^n - v^*) \leqslant$$
$$\|F_v'(v^n,\cdot)\|\,\|v^{n+1} - v^n\| + \frac{1}{h^n}\,\|v^{n+1} - v^n\|\,\|v^* - v^{n-1}\| \leqslant$$
$$\left(\|F_v'(v^n,\cdot)\| + \frac{1}{\varepsilon_1}\|v^* - v^{n+1}\|\right)\|v^{n+1} - v^n\|$$

Since the expression in parentheses is bounded and, as previously proved, $\|v^{n+1} - v^n\| \to 0$, hence $F(v^n,\cdot) \to F(v^*,\cdot)$ when $n\to\infty$. The theorem is proved.

REFERENCES

1. Krasovskii, N. N., Theory of Motion Control. Moscow, "Nauka", 1968.
2. Kozhevnikov, Iu. V., On the principle of optimality in the mean for discontinuous stochastic systems. Avtomatika i Telemekhanika, No. 10, 1966.
3. Gikhman, I. I. and Skorokhod, A. V., Introduction to the Theory of Random Processes. Moscow, "Nauka", 1965.
4. Rozenberg, G. S., Sufficient conditions of optimality for dynamic systems defined by stochastic differential equations. Avtomatika i Telemekhanika, No. 12, 1970.
5. Krasovskii, A. A., Phase Space and Statistical Theory of Dynamic Systems. Moscow, "Nauka", 1974.

6. M e r k l i n g e r, K. J., Numerical analysis of nonlinear control systems using the Fokker –Plank –Kolmogorov equations. Trans. 2-nd Inern. Federation on Automatic Control. Basel, 1963. Moscow, "Nauka", 1965.

7. B u d a k, B. M. and B e r k o v i c h, E. M., On the approximation of extremal problems. Zh. Vychisl. (English translation), Pergamon Press, J. USSR Comput. Mat. mat. Physics, Vol. 11, No. 3, 1971.

8. L e v i t i n, E. S. and P o l i a k, B. T., Method of minimization in the presence of constraints. (English translation), Pergamon Press, J. USSR Comput. Mat. mat. Physics, Vol. 6, No. 5, 1966.

9. I o f f e, L. D. and T i k h o m i r o v, V. M., Theory of Extremal Problems. Moscow, "Nauka", 1974.

10. P o n t r i a g i n, L. S., B o l t i a n s k i i, V. G., G a m k r e l i d z e, R. V., and M i s h c h e n k o, E. F., The Mathematical Theory of Optimal Processes. (English translation), Pergamon Press, Book No. 10176, 1964.

Translated by J.J. D.